Introduction to cyclic homology

Axel GASTALDI*

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This lecture tends to propose an introduction of cyclic homology as a homological approximation of K-theory. Initially introduced by Rinehart and Connes as a modification of Hochschild homology implementing a noncommutative analogue of de Rham theory, this invariant turns out to produce interesting K-theory traces. Indeed, given any even cyclic cocycle ϕ over a complex algebra A, the assignment $e \mapsto \phi(e, \cdots, e)$ defines an additive morphism $K_0(A) \to \mathbb{C}$. This construction, known as the Chern-Connes pairing between K-theory and even periodic cyclic cohomology, exhausts all possible index pairings. Its computation has been performed in many cases and provides both explicit formulas of topological invariants and interesting results in representation-theory. The periodic cyclic homology being a finer invaraint than K-theory, this pairing is mostly irrelevant for the $C^\star$-algebraic case but becomes informative after replacement by a *good* subalgebra. The entire and asymptotic cyclic homology overcome these problem by proposing coarser invariants well-adapted to the $C^\star$-algebraic framework.

1 Hochschild homology

The Hochschild homology arose naturally in the algebraic theory as a machine to compute the failure of commutativity of an algebra. This homology defines a non-commutative generalization of the notion of differential forms. This analogy is known as the Hochschild-Kostant-Rosenberg theorem and propelled the domain of non-commutative Geometry in the early 60's. A complete introduction is given in [Lod92, p. 1.1].

Let A be a complex unital algebra. We use $\otimes$ to denote the tensor product over the complex numbers. For $n \geq 0$, consider the vector space $C_n(A) := A^{\otimes n+1}$. One can define $n+1$ operators denoted $d_i : C_n(A) \to C_{n-1}(A)$, for $0 \leq i \leq n$, given by:

$$d_i(a_0 \otimes \cdots \otimes a_n) := \begin{cases} a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n & i = 0 \\ a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}$$

These operators verify the identities $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$, then induce a differential map

$$b := \sum_{i=0}^{n} (-1)^i d_i : C_n(A) \to C_{n-1}(A). \tag{1}$$

verifying the condition $b \circ b = 0$.

Definition 1.1. *We define the* ***Hochschild homology of a complex unital algebra*** *A as the complex vector spaces*

$$HH_\star(A) := H_\star(C_\star(A), b).$$

*Research Center for Operator Algebras, ECNU

The Hochschild homology groups are functorial for the variable in the sense that any homomorphism of complex unital algebras $f : A \to B$ induces a linear map $HH(f) : HH_\star(A) \to HH_\star(B)$ at any degree.

Exercise 1 Show that for any unital algebra A, there exists an isomorphism $HH_0(A) \simeq A/[A, A]$ where $[A, A] = \langle ab - ba \mid a, b \in A \rangle$ denotes the linear-span of commutators of the algebra. Conclude that $HH_0(A) \simeq A$ if and only if A is commutative.

The Hoshchild homology can be defined using coefficients. The coefficient space requires a right and left actions of A to ensure the differential b is well-defined. Such a space is called a bimodule over A. We define the Hochschild homology of a complex algebra A with **coefficients in a A-bimodule** M as the homology of the chain complex made of the chains $C_n(A, M) := M \otimes A^{\otimes n}$ and the differential b as before. Its homology groups are denoted $HH_\star(A, M)$ and functorial in both variables. We recover $HH_\star(A)$ as the Hochschild homology $HH_\star(A, A)$ with coefficients in the A-bimodule A itself. One can find this construction in many book about homological algebra.

Remark There exists a standard way to extend a functor R from unital algebras with values in vector space to the category of not necessarily unital algebras. The idea is the following. Given a complex *non-unital* algebra A one can adjoined to it a unit by considering the algebra $A_+ := \mathbb{C} \oplus A$ endowed with the product $(\lambda, a) \cdot (\lambda', a') = (\lambda\lambda', \lambda a' + a\lambda' + aa')$. By definition, the extension of R to non necessarily unital algebra is given by

$$R(A) := \mathrm{Coker}(R(\mathbb{C}) \longrightarrow R(A_+)) \tag{2}$$

which is well defined by functoriality. We will then mainly define the homologies we will encounter for unital algebras and use this process to extends their definition to the category of non necessarily unital algebra.

Remark The Hochschild homology corresponds to the *derived functor* [GM02] of the left-exact functor $A \mapsto A/[A, A]$, or in other words to the Tor-functor $HH_\star(A) \simeq \mathrm{Tor}_\star^{A\text{-}\mathbf{bimod}}(A, A)$. This point of view allows us to compute this homology using other chain-complexes. Whenever $P_\star(A)$ is a projective resolution of A as a A-bimodule, its Hochschild homology is equal to $HH_\star(A) \simeq H_\star(A \otimes_{A\text{-}\mathbf{bimod}} P_\star(A))$ and the result doesn't depend on the resolution.

In the case of topological algebras we must specify which particular topological tensor product we use to compute Hochschild theory. From now to the end, we will mostly work with **complex complete locally convex algebras**. This class of algebras includes Banach algebras, $C^\star$-algebras, Fréchet algebras and many smooth function algebras as *e.g.* of Schwartz-type. In this setup, the completed projective tensor product preserves the underlying topologies unlike the complex tensor product, used until now. Given two Banach vector spaces E and F, their **completed projective tensor product** is denoted $E\widehat{\otimes}_\pi F$ and given by the completion of $E \otimes F$ with respect to the largest semi-norm $\|\cdot\|_\pi$ such that $\|e \otimes f\|_\pi = \|e\|_E \cdot \|f\|_F$ on elementary tensors [Gro54]. With two compact manifolds M and N, the completed projective tensor product realizes the identity

$$\mathcal{C}^\infty(M)\widehat{\otimes}_\pi\mathcal{C}^\infty(N) \simeq \mathcal{C}^\infty(M \times N).$$

Definition 1.2. *We define the **Hochschild homology of a complex locally convex algebra** A as $HH_\star(A) := H_\star(C_\star(A, \widehat{\otimes}_\pi), b)$, the homology of its Hochschild complex defined using complete projective tensor product.*

Let us compute some examples:

- $A = \mathbb{C}$: The zero-th homology groups has already been computed in exercise 1 and all the other homology groups are zero:

$$HH_n(\mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0; \\ 0 & \text{if } n > 0. \end{cases}$$

- $A = \mathbb{C}[T]$: Again by exercise 1, we know that $HH_0(\mathbb{C}[T]) \simeq \mathbb{C}[T]$. We can show that $HH_1(\mathbb{C}[T]) \simeq \mathbb{C}[T]$ and $HH_n(\mathbb{C}[T]) = 0$ for any $n > 1$. More generally, we have

$$HH_n(\mathbb{C}[T_1, \cdots, T_r]) = 0 \text{ for } n > r.$$

- $A = \mathbb{C}[\Gamma]$ is the convolution algebra of a finite group Γ, i.e. its space of Dirac measures with pointwise product. Its Hochschild homology recover the homology of the group [Mac63].

  Theorem 1.3 (Mac Lane)**.** *For every $\mathbb{C}[\Gamma]$-bimodule M, there exists an isomorphism*

$$HH_\star(\mathbb{C}[\Gamma], M) \simeq H_\star(\Gamma, \widetilde{M})$$

  where $\widetilde{M}$ is endowed with the G-action given by $g \cdot m := gmg^{-1}$.

- $A = \mathcal{C}^\infty(M, \mathbb{C})$ is the algebra of smooth functions over a compact manifold: The description of its Hochschild complex in terms of differential forms on M yields to the following important theorem [HKR09].

  Theorem 1.4. ✾ (Hochschild-Kostant-Rosenberg) *Let M be a compact manifold. The Hochschild homology of $\mathcal{C}^\infty(M) := \mathcal{C}^\infty(M, \mathbb{C})$ is isomorphic to the graded-algebra of differential forms over M*

$$HH_\star(\mathcal{C}^\infty(M)) \simeq \Omega^\star(M).$$

  Furthermore, the isomorphism is given by the anti-symmetrization map

$$\epsilon : f_0 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n. \tag{3}$$

2 Cyclic (co)homology

Although Hochschild homology plays the role of differential forms over a manifold, it lacks the important information of the De Rham differential. In 1963, G. Rinehart proposed a generalization of the De Rham cohomology which fits in the Hochschild-Kostant-Rosenberg theorem above [Rin63]. A. Connes in his book [Con85] defined an operator

$$HH_n(A) \xrightarrow{B} HH_{n+1}(A)$$

whose aim is to play the role of the De Rham differential in a non-commutative framework. Through the anti-symmetrization map ϵ defined in (3) above, it becomes a multiple of the exterior derivative on the De Rham complex

$$\begin{array}{ccc} HH_n(\mathcal{C}^\infty(M)) & \xrightarrow{B} & HH_{n+1}(\mathcal{C}^\infty(M)) \\ \epsilon \downarrow \sim & & \sim \downarrow \epsilon \\ \Omega^n(M) & \xrightarrow{(n+1)\cdot d} & \Omega^{n+1}(M) \end{array} \tag{4}$$

The differential b together with the operator B (which is a differential) gives naturally rise to the study of a bicomplex whose homology is called *cyclic homology*

and computes a non-commutative De Rham cohomology.

Fix A to be a unital (complete locally convex) algebra. The **Connes' boundary operator** $B : C_n(A) \longrightarrow C_{n+1}(A)$ is defined at the level of chain-complex as follows

$$B(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} \Big[(-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} - (-1)^{n(i-1)} a_{i-1} \otimes 1 \otimes a_i \otimes \cdots a_n \otimes a_0 \otimes \cdots \otimes a_{i-2} \Big].$$

Remark In [Lod92, p. 2.1.7] you can find an alternative definition of the Connes' boundary map using the cyclic operator $t : C_n(A) \to C_n(A)$ given by $t(a_0 \otimes \cdots a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$. It is the historical definition and justifies the term *"cyclic"* in the name cyclic homology.

Proposition 2.1. *The Connes' boundary map verifies the properties*

$$B \circ B = 0 \quad \textit{and} \quad Bb + bB = 0.$$

The chain-complex $C_\star(A)$ is endowed with the two commuting differentials b and B of degree -1 and +1 respectively, which yields to a chain bicomplex. Let us call **cyclic bicomplex** of a unital (complete locally convex) algebra the following

$$\mathcal{B}_\star(A) := \begin{array}{ccccccc} & \downarrow b & & \downarrow b & & \downarrow b & \\ C_2(A) & \xleftarrow{B} & C_1(A) & \xleftarrow{B} & C_0(A) & \xleftarrow{B} & \\ \downarrow b & & \downarrow b & & & & \\ C_1(A) & \xleftarrow{B} & C_0(A) & & & & \\ \downarrow b & & & & & & \\ C_0(A) & & & & & & \end{array} .$$

Computation of the homology of a bicomplex One can associate to any bicomplex $(C_{\star,\star}, \partial, \partial')$ a chain-complex Tot $C_{\star,\star}$ called **total complex**. Its n-chains are given by $(\text{Tot } C_{\star,\star})_n := \bigoplus_{p,q=n} C_{p,q}$ and it is endowed with the differential $\partial + \partial'$. It can be viewed as a *diagonal* complex where you glue together the successive diagonals.

$$C_{\star,\star} = \begin{array}{ccccccc} & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\ C_{0,2} & \xleftarrow{\partial'} & C_{1,2} & \xleftarrow{\partial'} & C_{2,2} & \xleftarrow{\partial'} & \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ C_{0,1} & \xleftarrow{\partial'} & C_{1,1} & \xleftarrow{\partial'} & C_{2,1} & \xleftarrow{\partial'} & \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ C_{0,0} & \xleftarrow{\partial'} & C_{1,0} & \xleftarrow{\partial'} & C_{2,0} & \xleftarrow{\partial'} & \end{array}$$

$$\text{Tot } C_{\star,\star} = C_{0,0} \xleftarrow{\partial+\partial'} \bigoplus_{p+q=1} C_{p,q} \xleftarrow{\partial+\partial'} \bigoplus_{p+q=2} C_{p,q} \xleftarrow{\partial+\partial'} \cdots$$

Definition 2.2. *The **cyclic complex** of a unital (complete locally convex) algebra A is defined to be the total complex of the cyclic bicomplex, denoted $CC_\star(A) :=$*

Tot $\mathcal{B}_{\star,\star}(A)$. *Its homology is called* ***cyclic homology*** *of* A*:*

$$HC_\star(A) := H_\star(CC_\star(A)).$$

When A *is non-unital, we use the trick (2) to define its cyclic homology.*

Exercise 2 For any (complete locally convex) algebra A, the zeroth cyclic homology group is always $HC_0(A) \simeq HH_0(A) \simeq A/[A, A]$.

Remark One can wonder what *cyclic* homology has to do with cyclicity. In the early works of Connes [Con85], he adopted another definition of cyclic homology which turns out to be equivalent to the one we proposed. Given a unital (complete locally convex) algebra A, we can consider the **cyclic operator** $t : C_n(A) \longrightarrow C_n(A)$ on its chains, given by $t(a_0 \otimes \cdots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$. Of course, this operator commutes with the Hochschild operator b and one can define the chain-complex $C^\lambda_\star(A) := C_\star(A)/(1-t)(C_\star(A))$ with differential b. It can be thought as the covariant complex of the action of the cyclic group. When the ground field of A is of characteristic zero, one obtains the isomorphism

$$HC_\star(A) \xrightarrow{\sim} H_\star(C^\lambda_\star(A), b)$$

given by the projection $C_n(A) \longrightarrow C^\lambda_n(A)$ on the first column of $\mathcal{B}_\star(A)$ and zero elsewhere.

A way to compute cyclic homology is to use a long exact sequence involving both Hochschild and cyclic homology groups. The first column of the bicomplex $(\mathcal{B}_{\star,\star}(A), b, B)$ corresponds exactly to the chain-complex $(C_\star(A), b)$, computing the Hochschild homology. The image of the embedding $I : C_\star(A) \to \mathcal{B}_\star(A)$ then corresponds to the kernel of the two-degree shift $S : \mathcal{B}_\star(A) \to \mathcal{B}_\star(A)[-2]$ called **periodicity map**. The following sequence of complexes is then exact:

$$0 \longrightarrow C_\star(A) \xrightarrow{I} \mathcal{B}_\star(A) \xrightarrow{S} \mathcal{B}_\star(A)[-2] \longrightarrow 0. \tag{5}$$

By an argument of a long exact sequence we obtain the following statement we can find in [Con85].

Proposition 2.3 (Connes' exact sequence)**.** *Given any unital (complete locally convex) algebra* A *over a field of characteristic zero, the following sequence is exact for all* $n \geq 0$

$$HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A)$$

Corollary 2.4. *Any (continuous) morphism* $f : A \to B$ *of (complete locally convex) algebras such that* $HH(f) : HH_\star(A) \to HH_\star(B)$ *is an isomorphism in Hochschild homology induces an isomorphism* $HC(f) : HC_\star(A) \to HC_\star(B)$ *in cyclic homology. The converse is also holds.*

Proof. By induction on the homological degree using the Connes' exact sequence, the initialization being a consequence of the fact that $HH_0(A) \simeq HC_0(A)$. □

Let us compute some examples:

- Let A be a (complete locally convex) algebra such that $HH_n(A) = 0$ for $n > 0$ (for instance $A = \mathbb{C}$). The exact sequence above shows that

$$HC_n(A) \simeq \begin{cases} HH_0(A) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

- $A = \mathcal{C}^\infty(M, \mathbb{C})$ is the algebra of smooth functions over a compact manifold: The bicomplex $CC_\star(\mathcal{C}^\infty(M))$ encapsulated both the data of differentials forms and of the exterior derivative. The observation of G. Rinehart in the early 60's is that the homology of its total complex needs to be related to the De Rham cohomology, which is the following theorem [Lod92, p. 2.3.7].

  Theorem 2.5. *If M is a compact manifold, the cyclic homology of its algebra of smooth functions $\mathcal{C}^\infty(M)$ can be describe with De Rham cohomology groups*

$$HC_n(\mathcal{C}^\infty(M)) \simeq \Omega^n(M)/d(\Omega^{n-1}(M)) \oplus H^{n-2}_{dR}(M;\mathbb{C}) \oplus H^{n-4}_{dR}(M;\mathbb{C}) \oplus \cdots$$

Exercise 3 In a straightforward computation, show that the diagram (4) commutes when M is a compact manifold (*i.e.* $\mathcal{C}^\infty(M)$ is unital). Remember that in the exterior algebra, the terms can be reorganized by multiplying with the sign of the permutation.

The periodicity map According to the Hochschild-Kostant-Rosenberg theorem and the exercise above, the inclusion of the quotient space $\Omega^n(M)/d(\Omega^{n-1}(M))$ into $HC_n(\mathcal{C}^\infty(M))$ for any $n \geq 0$ yields the following comparison of exact sequences.

$$\begin{array}{ccccccc}
\Omega^n(M) & \xrightarrow{\pi} & \Omega^n(M)/d(\Omega^{n-1}(M)) & \xrightarrow{0} & \Omega^{n-2}(M)/d(\Omega^{n-3}(M)) & \xrightarrow{d} & \Omega^{n-1}(M) \\
\sim\downarrow & & \downarrow & & \downarrow & & \downarrow\sim \\
HH_n(\mathcal{C}^\infty(M)) & \xrightarrow[I]{} & HC_n(\mathcal{C}^\infty(M)) & \xrightarrow[S]{} & HC_{n-2}(\mathcal{C}^\infty(M)) & \xrightarrow[B]{} & HH_{n-1}(\mathcal{C}^\infty(M))
\end{array}.$$

Let me pinpoint the fact that the periodicity map is zero at the level of De Rham cohomology, which enrich the cyclic homology with an intricate periodic structure. It will play an important regarding the relation between successive Chern classes that we will see in lemma 3.5.

The periodicity map in the Connes' exact sequence can be also thought as the a Gysin-morphism in homology, coming from a S^1-sphere bundle. To make the link, Connes defined a class of spaces called cyclic spaces. A **cyclic space** X is, according to [Con85, Appendix A], a simplicial space together with a family of homeomorphisms $t_n : X_n \to X_n$ verifying $t_n^{n+1} = id$ and such that for any $i > 0$

$$\partial_i t_n = t_{n-1}\partial_{i-1}, \quad \partial_0 t_n = \partial_n, \quad \sigma_i t_n = t_{n+1}\sigma_{i-1} \text{ and } \sigma_0 t_n = t_{n+1}^2 \sigma_n$$

where ∂_i are boundaries and σ_i are degeneracies of X. One can associate to a cyclic space X two topological spaces $|X|$ and $|X|^{\text{cy}}$ called **geometric realization** and **cyclic geometric realization** of X such that there exists a S^1-principal bundle $\pi : |X| \longrightarrow |X|^{\text{cy}}$. Moreover, for any cyclic space X there exists a complex algebra $\mathbb{C}[X]$ called **cyclic module** of X such that one has isomorphisms in complex singular homology for all $n \geq 0$

$$HH_n(\mathbb{C}[X]) \simeq H_n(|X|) \quad \text{and} \quad HC_n(\mathbb{C}[X]) \simeq H_n(|X|^{\text{cy}})$$

Through these isomorphisms, the Gysin morphism associates to S^1-principal bundle obtained as cap-product by the Euler class $e \in H^2(|X|^{\text{cy}})$ is equal to the periodicity map, and the following diagram commutes

$$\begin{array}{ccccccc}
HH_n(\mathbb{C}[X]) & \xrightarrow{I} & HC_n(\mathbb{C}[X]) & \xrightarrow{S} & HC_{n-2}(\mathbb{C}[X]) & \xrightarrow{B} & HH_{n-1}(\mathbb{C}[X]) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
H_n(|X|) & \xrightarrow[\pi^\star]{} & H_n(|X|^{\text{cy}}) & \xrightarrow[e\cap -]{} & H_{n-2}(|X|^{\text{cy}}) & \xrightarrow[\pi_\star]{} & H_{n-1}(|X|)
\end{array}$$

where $\pi^\star$ and $\pi_\star$ denote respectively the pull-back and the push-forward (or integration along fibers) of the bundle.

The group algebra An example of algebra whose cyclic homology is interesting to compute is the group algebra $\mathbb{C}[\Gamma]$ associated to a discrete group Γ. We fix an element $x \in \Gamma$ and we write $\langle x \rangle$ for its conjugacy class. For all n, we can define the family spent by:

$$\{\delta_{g_0} \otimes \cdots \otimes \delta_{g_n} \mid g_i \in \Gamma \text{ and } g_0 \cdots g_n \in \langle x \rangle\} \subseteq \mathbb{C}[\Gamma]^{\otimes n+1}.$$

It is stable under the differential b and t because these operators are just intern multiplications and permutations. We write $C_\star(\mathbb{C}[\Gamma])_x$ and $C^\lambda_\star(\mathbb{C}[\Gamma])_x$ for the subcomplexes of $C_\star(\mathbb{C}[\Gamma])$ and $C^\lambda_\star(\mathbb{C}[\Gamma])$ generated by these families and $HH_\star(\mathbb{C}[\Gamma])_x$ and $HC_\star(\mathbb{C}[\Gamma])_x$ for their contribution in Hochschild homology and cyclic homology.

Proposition 2.6. *The Hochschild complex and cyclic complex of the group algebra $\mathbb{C}[\Gamma]$ decompose along the space of conjugacy classes $\langle G \rangle$ of G and so the associated homology groups*

$$HH_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x \rangle \in \langle G \rangle} HH_\star(\mathbb{C}[\Gamma])_x \quad \text{and} \quad HC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x \rangle \in \langle G \rangle} HC_\star(\mathbb{C}[\Gamma])_x.$$

One may expect to compute the homology of the *local* complexes to obtain information on the *global* complex. First, recall that the **classifying space** $B\Gamma$ is a discrete group is a topological space such that $\pi_1(B\Gamma) \simeq \Gamma$ and whose complex singular homology corresponds to the group homology of Γ, namely $H_\star(B\Gamma) \simeq H_\star(\Gamma, \mathbb{C})$. Fix $x \in \Gamma$ and call $\Gamma_x := \{g \in \Gamma \mid gx = xg\}$ the centralizer of x in Γ. It is equipped with an $\mathbb{Z}$-action $\gamma_x : \mathbb{Z} \times \Gamma_x \to \Gamma_x$ given by $\gamma_x(n, g) := x^n g$ which induces a S^1-action $B\gamma_x : S^1 \times B\Gamma_x \to B\Gamma_x$ on its classifying space. The following proposition gives a description of the local components of the cyclic and Hochschild homologies of the group algebra.

Proposition 2.7. *For any $x \in \Gamma$, the Hochschild homology and cyclic homology of the group algebra $\mathbb{C}[\Gamma]$ over x can be computed as singular and S^1-equivariant singular homologies of the associated classifying space $B\Gamma_x$, namely*

$$H_\star(B\Gamma_x) \simeq HH_\star(\mathbb{C}[\Gamma])_x \quad \text{and} \quad H^{S^1}_\star(B\Gamma_x) \simeq HC_\star(\mathbb{C}[\Gamma])_x$$

via the morphism of chain-complexes $(g_0, \cdots, g_n) \mapsto \delta_{(g_1 \cdots g_n)^{-1}} \otimes \delta_{g_1} \otimes \cdots \delta_{g_n}$.

When x is of infinite order, the S^1-equivariant homology of the classifying space $B\Gamma_x$ is modeled by the singular homology of the the classifying space of the quotient group $B(\Gamma_x/(x))$. It precise the previous description as follows.

Theorem 2.8. *The Hochschild homology and cyclic homology of the group algebra $\mathbb{C}[\Gamma]$ can be computed as follows (see [Lod92, §7.4.10] and [Bur85, Theorem I]):*

$$HH_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x \rangle \in \langle G \rangle} H_\star(\Gamma_x, \mathbb{C}),$$

$$HC_\star(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle x \rangle \in \langle G \rangle^{\text{fin}}} H^{S^1}_\star(B\Gamma_x) \oplus \bigoplus_{\langle x \rangle \in \langle G \rangle^{\infty}} H_\star(\Gamma_x/(x), \mathbb{C}),$$

where the superscripts "fin" et "∞" split the cases wether x is of finite or infinite order in Γ.

Theorem 2.9. *The Hochschild and cyclic homologies verify the following fundamental properties:*

- *(Morita invariance)* [Lod92, pp. 1.2.4, 2.2.9] *For any (complete locally convex) algebra A and $r \geq 0$, the trace map induces isomorphisms*

$$HH_\star(M_r(A)) \simeq HH_\star(A) \quad \text{and} \quad HC_\star(M_r(A)) \simeq HC_\star(A).$$

- *(Unital Excision)* [Lod92, pp. 1.4.10, 2.2.17] *Let* $0 \to I \to A \to B \to 0$ *be an extension of complete locally convex with continuous linear section and* A *unital, the following sequences are exact for all* $n \geq 0$

$$HH_n(I) \longrightarrow HH_n(A) \longrightarrow HH_n(B) \longrightarrow HH_{n-1}(I)$$

.

$$HC_n(I) \longrightarrow HC_n(A) \longrightarrow HC_n(B) \longrightarrow HC_{n-1}(I)$$

Why cyclic homology is not the best homology? Consider the topological space $\mathbb{R}^2$. It is smoothly contractible to the point so $\mathcal{C}^\infty(\mathbb{R}^2)$ and $\mathbb{C}$ are smoothly homotopy equivalent algebras. Surprisingly, their cyclic homology groups aren't isomorphic. Indeed, our previous computations show that

$$HC_1(\mathcal{C}^\infty(\mathbb{R}^2)) \simeq \Omega^1(\mathbb{R}^2)/d(\Omega^0(\mathbb{R}^2)) \quad \text{and} \quad HC_1(\mathbb{C}) = 0,$$

the first quotient space being non-trivial because the 1-form $\omega = xdy$ is not closed, then doesn't lie in the image of the exterior differential. Namely, one must remember that

Cyclic homology is not stable under smooth homotopy!

The result [Tab11] of G. Tabuado states that any homology theory which is *Morita invariant*, *satisfies excision* and *stable under smooth homotopy* must be $\mathbb{Z}/2\mathbb{Z}$-graded. This is the reason why we introduce the periodic cyclic homology, a $\mathbb{Z}/2\mathbb{Z}$-graded homology that respect all these properties and plays a relevant role for the study of K-theory.

3 Periodic cyclic homology

The K-theory is a fundamental tool in non-commutative geometry [Kar78]. Introduced in the 60's by Alexander Grothendieck in order to states the well known Grothendieck-Riemann-Roch theorem, it kept developing to appear as a major tool in the proofs of Bott periodicity and Atiyah-Singer theorems. A big issue of this theory is that the computation of these groups are highly non-trivial. The idea behind periodic cyclic homology is to propose a more computable homology verifying powerful properties and endowed with a pairing with K-theory.

We fix A to be a complete locally convex algebra as before. We can define the differentials $b : C_n(A) \to C_{n-1}(A)$ and $B : C_n(A) \to C_{n+1}(A)$ on the graded vector space $C_\star(A)$ as above. These operators verify the property:

$$b^2 = B^2 = (b + B)^2 = 0.$$

We splits the Hochschild complex into the *even* and *odd* parts:

$$C_{\text{even}}(A) := \prod_{n\geq 0} C_{2n}(A) \quad \text{and} \quad C_{\text{odd}}(A) := \prod_{n\geq 0} C_{2n+1}(A).$$

The operator $b+B$ sends any even (*resp.* odd) degree element to a pair of odd (*resp.* even) degree elements, thus defines a differential map on the following $\mathbb{Z}/2\mathbb{Z}$-graded complex.

Definition 3.1. *We call* ***periodic cyclic complex of*** A *the following complex:*

$$\widehat{CC}_\bullet(A) := \; C_{\text{even}}(A) \underset{b+B}{\overset{b+B}{\rightleftarrows}} C_{\text{odd}}(A)$$

The periodic cyclic complex has only two homology groups, the even and the odd, often called zeroth and first. We will distinguish the different type of graduations using the subscript "$\bullet$" for any $\mathbb{Z}/2\mathbb{Z}$-theories. Also, one might remark that the notation to designate the periodic cyclic complex corresponds to the notation for the cyclic complex with a hat. Indeed, it comes from a more general construction which associated to a $\mathbb{Z}$-graded complex its periodic $\mathbb{Z}/2\mathbb{Z}$-graded complex, here $\widehat{CC}_\bullet(A)$ is the periodization of $CC_\star(A)$.

Definition 3.2. *We define the* ***periodic cyclic homology of*** *A as the homology of the periodic cyclic complex*

$$HP_\bullet(A) := H_\bullet(\widehat{CC}_\bullet(A)).$$

Exercise 4 Using exercise 2 show that if a (complete locally convex) algebra A verifies $HH_i(A) = 0$ for $i > 0$, then $HP_1(A) = 0$.

Theorem 3.3. ✾ *The periodic cyclic homology respects the following properties:*

- *(Morita invariance)* [LQ84] *For any (complete locally convex) algebra A and $r \geq 0$, the trace map induces an isomorphism*

$$HP_\bullet(M_r(A)) \simeq HP_\bullet(A).$$

- *(Excision)* [CQ97] [Pus01] *Let $0 \to I \to A \to B \to 0$ be an extension of complete locally convex with continuous linear section, the following is exact*

$$\begin{array}{ccccc} HP_0(I) & \longrightarrow & HP_0(A) & \longrightarrow & HP_0(B) \\ \uparrow & & & & \downarrow \\ HP_1(B) & \longleftarrow & HP_1(A) & \longleftarrow & HP_1(I) \end{array}$$

- *(Smooth homotopy invariance)* [Kas87] *If $f_0, f_1 : A \to B$ are continuous morphisms of locally convex algebras, homotopic via a smooth family of continuous morphisms $f_t : A \to B$, then $(f_0)_\bullet = (f_1)_\bullet : HP_\bullet(A) \longrightarrow HP_\bullet(B)$. Moreover, if M is a compact manifold, the antisymetrization map ϵ induces an isomorphism between periodic cyclic homology of its algebra of smooth functions $\mathcal{C}^\infty(M)$ and its De Rham cohomology groups*

$$HP_\bullet(\mathcal{C}^\infty(M)) \simeq \prod_{n\geq 0} H_{dR}^{2n+\bullet}(M;\mathbb{C}).$$

In geometry, the **Chern classes** are topological invariants (sometimes complete) which furnish a way to distinguishing vector bundles over a smooth manifold. They all live in even degree De Rham cohomology. The **Chern character** is a character that gathers these data, associating to a vector bundle over a given smooth manifold a polynomial with coefficients in its Chern classes [Hir56]. It factorizes over the even K-group ring of the manifold as is transforms tensor product of vector bundles into cohomological product of their Chern classes

$$\mathrm{Ch}_{dR} : K^0(M) \longrightarrow \prod_{n\geq 0} H_{dR}^{2n}(M;\mathbb{C}).$$

The left hand space is an abelian group while the right hand one is a complex vector space. These are difficult to compare but while tensoring over the rational numbers, the following description occurs [Ati19].

Theorem 3.4. ✾ (Atiyah-Grothendieck-Hirzebruch) *The Chern character is a rational isomorphism*

$$\mathrm{Ch}_{dR} \otimes_{\mathbb{Z}} \mathbb{C} : K^0(M) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \prod_{n \geq 0} H^{2n}_{dR}(M; \mathbb{C}).$$

As seen in the third point of theorem 3.3, the periodic cyclic homology of a unital algebra of smooth functions corresponds to the even and odd parts of the De Rham cohomology of its spectrum. This motivates the existence of an *algebraic version* of the Chern character from the K-theory of an algebra to its periodic cyclic homology. This would produce an algebraic computation for these unreachable K-groups.

Consider A to be a locally convex algebra. We recall that the set $\mathfrak{p}(A)$ of any-size idempotent matrices over A defines a monoid for the Whitney sum. The **even K-theory** of A is defined to be the Grothendieck group of this monoid

$$K_0(A) := \mathrm{Gr}(\mathfrak{p}(A)).$$

It is generated by differences $[e] - [f] \in K_0(A)$ where e and f belongs to $\mathfrak{p}(A)$. Given an idempotent matrix $e \in M_r(A)$ and $n \geq 0$, let us write

$$c_n(e) := \begin{cases} e & \text{if } n = 0 \\ (-1)^n \frac{(2n)!}{n!} \left(e - \frac{1}{2}\right) \otimes e^{\otimes 2n} & \text{otherwise} \end{cases} \in C_{2n}(M_r(A)).$$

The coefficient in front of this definition appears in the expression of the trace of an exponential matrix, the usual way to define the geometric Chern character with value in De Rham cohomology.

Lemma 3.5. *For any idempotent matrix $e \in M_r(A)$ and $n \geq 0$ the chain $c_n(e) \in C_{2n}(M_r(A))$ verifies $(b+B)(c_n(e)) = 0$ and defines the n-th Chern class of e*

$$\mathrm{Ch}_n(e) := [\mathrm{Tr}(c_n(e))] \in HC_{2n}(A)$$

which only depends on the class of the idempotent matrix. Moreover, one has the formula $S(\mathrm{Ch}_n(e)) = \mathrm{Ch}_{n-1}(e)$.

Definition 3.6. *The **Chern character of a locally convex algebra** A is a group morphism defined as*

$$\begin{array}{rccc} \mathrm{Ch}: & K_0(A) & \longrightarrow & HP_0(A) \\ & [e] & \longrightarrow & \mathrm{Ch}(e) := (c_0(e), c_1(e), \cdots) \end{array},$$

well-defined by Morita invariance.

This construction coincide with the geometric Chern character in the case of a smooth commutative algebra [Lod92, p. 8.3.9], namely $\mathrm{Ch}_{\mathrm{dR}} = \epsilon \circ \mathrm{Ch}$.

Although the Chern character is defined for a broad class of algebras, its existence does not guarantee that it provides useful information about their K-theory! The classes of algebras for which K-theory is well behaved and those for which periodic cyclic homology is informative do not necessarily coincide. For instance, consider a compact manifold M. On one hand, the continuous inclusion of its Fréchet algebra of smooth functions $\mathcal{C}^\infty(M)$ into its $C^\star$-algebra of continuous functions $\mathcal{C}^0(M)$ realizes an isomorphism in K-theory

$$K_0(\mathcal{C}^\infty(M)) \xrightarrow{\sim} K_0(\mathcal{C}^0(M)).$$

This is due to the De Rham theorem that relates the De Rham cohomology and the singular cohomology by integration [BT82]. On the other hand, whereas the

periodic cyclic homology of $\mathcal{C}^\infty(M)$ is isomorphic to the De Rham cohomology of the manifold (see theorem 3.3), its $C^\star$-algebra of continuous functions verifies

$$HP_0(\mathcal{C}^0(M)) \simeq \mathcal{B}(M) \quad \text{and} \quad HP_1(\mathcal{C}^0(M)) = 0$$

where $\mathcal{B}(M)$ is its space of Borel measures over M [Haa83]. While K-theory remains stable under these different completions, periodic cyclic theory drastically differentiate them

$$\begin{array}{ccc} K_0(\mathcal{C}^\infty(M)) & \xrightarrow{\sim} & K_0(\mathcal{C}^0(M)) \\ {\scriptstyle \mathrm{Ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ch}} \\ HP_0(\mathcal{C}^\infty(M)) & \xrightarrow{\sim} & HP_0(\mathcal{C}^0(M)) \end{array} .$$

This example shows how periodic cyclic homology and K-theory can behave quite differently in certain situations and that the Chern character from K-theory to periodic cyclic homology can be far from being an isomorphism. To go further, the introduction of the book [Pus03] cite various examples of the Chern character's behavior depending on the completion-type of the algebra.

However, in some cases the Chern character becomes a powerful tool but after replacing the underlying algebra A by another one $\mathcal{A}$ with same K-groups but better computable periodic cyclic homology

$$\mathrm{Ch} : K_0(A) \dashrightarrow K_0(\mathcal{A}) \longrightarrow HP_0(\mathcal{A}).$$

Since K-theory is stable under passage to dense subalgebras which are closed under holomorphic functional calculus (it is not the case for periodic cyclic homology), the main type of algebra one wants to work with are intermediate algebras between smooth and $C^\star$-algebras, *e.g.* of Schwartz or Hecke types. They simultaneously retain and reflect information in the K-theoretical side and possesses enough analytic structure to be computable in periodic cyclic homology.

We list some examples where such strategy provides some informations.

Theorem 3.7 (Baum-Higson-Lafforgue-Plymen-Solleveld)**.** [Sol05] [Sol07] [BHP00] [Laf06] *Let G be a reductive p-adic group. The Baum-Connes assembly map*

$$\mu : K_G^0(\mathcal{G}) \longrightarrow K_0(C_r^\star(G))$$

from the equivariant K-theory of its Bruhat-Tits building $\mathcal{G}$ to the K-theory of its reduced $C^\star$-algebra is rationally an isomorphism.

Proof. (Just the main strategy). Write $\mathcal{H}(G)$ and $\mathcal{S}(G)$ for the Hecke and Schwartz algebras of the reductive p-adic group G. The proof follows from the fact that the following diagram commutes

$$\begin{array}{ccccc} K_0(\mathcal{H}(G)) \simeq K_G^0(\mathcal{G}) & \xrightarrow{\mu} & K_0(C_r^\star(G)) \simeq K_0(\mathcal{S}(G)) \\ {\scriptstyle \mathrm{Ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ch}} \\ HP_0(\mathcal{H}(G)) & \xrightarrow{\quad\iota\quad} & HP_0(\mathcal{S}(G)) \end{array}$$

where the vertical Chern characters are rational isomorphisms and the bottom arrow is the continuous inclusion of the Hecke algebra into the Schwartz algebra, which is an isomorphism because these finite-type algebras are stratified equivalent. □

Theorem 3.8 (G., 2026)**.** *Given a real reductive Lie group G with maximal compact subgroup K, there exists a Chern character*

$$\mathrm{Ch} : K^\bullet(\mathcal{C}_\tau(G/K)) \simeq K^\bullet(\mathcal{C}_\tau^\infty(G/K)) \longrightarrow HP^\bullet(\mathcal{C}_\tau^\infty(G/K)) \simeq HP^\bullet(\mathcal{C}_c^\infty(G/K))$$

compatible with the left-translation action of the group G. Moreover, the product by the Chern character of the Dirac element $\not{D} \in K_G^{\dim(G/K)}(\mathcal{C}_\tau(G/K))$ realizes the degree-shift isomorphism

$$\mathrm{Ch}(\not{D}) \times - : HP_\bullet((A\widehat{\otimes}_\pi\mathcal{D}) \rtimes K) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}((A\widehat{\otimes}_\pi\mathcal{D}) \rtimes G)$$

for every locally convex algebra A endowed with a smooth action of G, after stabilization by a smooth Schatten ideal $\mathcal{D}$.

The pairing of K-theory with periodic cyclic cohomology Given a complete locally convex algebra A and $n \geq 0$, we write $C^n(A) := \mathcal{H}om(C_n(A), \mathbb{C})$ to denoted the continuous linear dual of the chain complex of A. It is made of continuous functions $\phi_n : A \times \cdots \times A \longrightarrow \mathbb{C}$ that are linear. The **periodic cyclic complex** of a complete locally convex algebra A is defined as

$$\widehat{CC^\bullet}(A) := \ C^{\mathrm{even}}(A) \underset{(b+B)^\vee}{\overset{(b+B)^\vee}{\rightleftarrows}} C^{\mathrm{odd}}(A)$$

where $(b^\vee + B^\vee)(\phi) := \phi \circ (b + B)$ is the dual of the periodic cyclic differential. An even or odd cochain is then a collection of continuous linear functions $(\phi_{2n})_n$ or $(\phi_{2n+1})_n$ with complex value. This complex yields a homology called **periodic cyclic cohomology** of A, given by $HP^\bullet(A) := H^\bullet(\widehat{CC^\bullet}(A))$. Periodic cyclic homology still respects Morita invariance, excision and smooth homotopy invariance in the sense of theorem 3.3. Moreover, in many cases, periodic cyclic cohomology verifies

$$HP^\bullet(A) := \lim_{\to S} HC^{2n+\bullet}(A).$$

Exercise 5 Shows that every trace τ over a complete locally convex algebra defines a class $[\tau] \in HP^0(A)$.

Remark By definition, one can always pair periodic cyclic cycles and periodic cyclic cocycles $\langle -, - \rangle : HP_0(A) \times HP^0(A) \to \mathbb{C}$ via the evaluation map. In particular, every cocycle $[\phi] \in HP^0(A)$ induces a linear map $\langle -, [\phi] \rangle : HP_0(A) \to \mathbb{C}$ which makes $HP^0(A)$ a linear subspace of $HP_0(A)^\vee$. When $A = \mathcal{C}^\infty(M)$ is the algebra of smooth functions over a compact manifold M of dimension m, the Hochschild-Kostant-Rosenberg theorem together with the Poincaré duality yield $HP^0(\mathcal{C}^\infty(M)) \subseteq H_{dR}^{m-\mathrm{even}}(M)$ and the evaluation pairing becomes in this case

$$\langle [\omega], [\omega'] \rangle = \int_M \omega' \wedge \omega \in \mathbb{C}$$

where $[\omega] \in HP_0(\mathcal{C}^\infty(M)) \simeq H_{dR}^{\mathrm{even}}(M)$ and $[\omega'] \in HP^0(\mathcal{C}^\infty(M)) \subseteq H_{dR}^{m-\mathrm{even}}(M)$.

Proposition 3.9. *Let A be a unital locally convex algebra. There exists a complex valued pairing $\langle -, - \rangle : K_0(A) \times HP^0(A) \longrightarrow \mathbb{C}$ called **Chern-Connes pairing** between K-theory and periodic cyclic cohomology, given by*

$$\langle [e], [\phi] \rangle := \phi(\mathrm{Ch}(e)) = \sum (-1)^n \frac{(2n)!}{n!} \phi_{2n}(e - 1/2, e, \cdots, e)$$

where e is an idempotent matrix over A and $\phi = (\phi_{2n})_n$ is an even periodic cyclic cochain of matrices with coefficients in A.

Remark To my knowledge, the Chern-Connes pairing takes its origin from a simple but fundamental observation provided by Connes in [Con85]. Let A be an algebra and $\phi : A \times \cdots \times A \longrightarrow \mathbb{C}$ be a linear map. If ϕ verifies, for all $a_i \in A$, the conditions

- $\phi(a_0, \cdots, a_n) = (-1)^n \phi(a_n, a_0, \cdots, a_{n-1})$;
- $\sum_{i=0}^{n} (-1)^i \phi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) = (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \cdots, a_n)$

then the assignment $e \mapsto \phi(e, \cdots, e)$ defines a morphism $K_0(A) \to \mathbb{C}$. These conditions are equivalent for ϕ to be cyclic and closed for the Hochschild differential, namely that ϕ defines a cyclic cocycle in $HC^{2n}(A)$.

One of the main important fact about this construction is that one can express Fredholm indexes using this Chern-Connes pairing [Con85, Chapter 7] [GE20].

Definition 3.10. *An operator T on a Hilbert space is said to be **p-summable** and we write $T \in \ell^p$ if its p-Schatten norm is finite, namely if $\mathrm{Tr}\,(T^\star T)^p < \infty$. A bounded Fredholm module $(\mathcal{H}, F, \pi)$ on a locally convex algebra A is said to be **finitely-summable** if there exists $p > 0$ such that the commutator $[F, \pi(a)]$ is p-summable for every $a \in A$.*

Theorem 3.11. ✾ (Connes-Moscovici) *Let A be a unital locally convex algebra and $(\mathcal{H}, F, \pi)$ a p-summable bounded Fredholm module over it with property $F^2 = 1$. For $2n \geq p$, the formula*

$$\phi_{2n}(a_0, \cdots, a_{2n}) := (-1)^n \mathrm{Tr}\Big(\gamma \pi(a_0)[F, \pi(a_1)] \cdots [F, \pi(a_{2n})]\Big)$$

where γ is the grading morphism defines an even periodic cyclic cocycle $\phi_F = (\phi_{2n})_n$ which depends only on the homotopy class of the finite-summable Fredholm module. Moreover, for any idempotent matrix e over A the analytic index of the Fredholm operator $F_e := \pi(e) F \pi(e)$ acting on $\mathcal{H}$ can be expressed as

$$\mathrm{Index}_{\mathrm{an}}(F_e) = \langle [e], [\phi_F] \rangle \in \mathbb{Z}.$$

Firstly, this theorem constructs a cocycle associated to a finite-summable Fredholm module (the assumption insures that the trace above is well-defined), which can be thought as a *partial Chern character* $\mathrm{Ch}^\vee$ from some K-homology classes to periodic cyclic cohomology. Secondly, it states that the integral pairing between K-theory and K-homology can be recovered as the evaluation pairing between periodic cyclic homology and cohomology

$$\langle -, - \rangle_K = \langle \mathrm{Ch}(-), \mathrm{Ch}^\vee(-) \rangle_{HP}.$$

Remark In the case of spin Dirac operators, one can recover the Atiyah-Singer index formula from the Connes-Moscovici index theorem. Take $A = \mathcal{C}^\infty(M)$, the algebra of smooth functions over a compact even-dimensional spin-manifold M. If E is a smooth vector bundle over M, its sections defines an idempotent matrix e over $\mathcal{C}^\infty(M)$ and we can compute $\mathrm{Ch}(e) = \mathrm{Ch}(E) \in H^{\mathrm{even}}_{dR}(M)$. Furthermore, if $\not{D}^+ : \Gamma(S^+) \to \Gamma(S^-)$ is the spin Dirac operator over M, its associated periodic cyclic cocycle corresponds to the $\widehat{A}$-genus $\widehat{A}(TM) \in HP^0(\mathcal{C}^\infty(M)) \subseteq H^{\mathrm{even}}_{dR}(M)$. Then, the Connes-Moscovici theorem asserts that the analytic index of the twisted spin Dirac operator $\not{D}^+_E : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$ can be computed as

$$\mathrm{Index}_{\mathrm{an}}(\not{D}^+_E) = \langle [E], [\phi_{\not{D}}] \rangle = \int_M \widehat{A}(TM) \wedge \mathrm{Ch}(E) = \mathrm{Index}_{\mathrm{top}}(\not{D}^+_E).$$

Analogous statement holds for any elliptic operator over M, involving principal symbols and Todd-classes [Con85] and [CM90].

Some explicit computations of the Chern-Connes pairing has been performed in various cases such as the smooth algebra of a compact manifold ([Con85] and

[CM90] as said before) and its equivariant case ([PT22]), the group algebra of a discrete group ([HW18] and [RWW20]), the Harish-Chandra's Schwartz algebra of a real reductive group ([ST19]), etc. Also, if G a real Lie group G and M a manifold equipped with a cocompact G-action, any elliptic operator D on a vector bundle over M produces an equivariant index

$$\mathrm{Index}_G(D) \in K_0(C_r^\star(G))$$

via the Baum-Connes assembly map. The computation of the different pairing of this index with periodic cyclic cocycles (such as orbital integrals) provides information about the Baum-Connes conjecture. The main computations have been performed by [HST20] and [HW17] using Harish-Chandra's Schwartz algebra and mostly rely on representation-theoretic information. This pairing both computes topological invariants (like Euler characteristic) and provides powerful representation theoretic statements (*e.g.* Selberg principle). Some results intimately related to the Baum-Connes conjecture, might rely on the computation of Chern-Connes pairings.

- (The idempotent Kadison conjecture) Suppose Γ is a torsion free group. Then, the only idempotent of $C_r^\star(\Gamma)$ are $e = 0, 1$.

  Proposition 3.12. *The canonical trace map $\tau : \mathbb{C}[\Gamma] \to \mathbb{C}$ given by*

  $$\tau\left(\sum a_g \delta_g\right) := a_e$$

  extends to a group morphism $\tau_\star : K_0(C_r^\star(\Gamma)) \longrightarrow \mathbb{C}$. If there exists a finitely-summable bounded Fredholm module $[(\mathcal{H}, F, \pi)] \in K^0(C_r^\star(\Gamma))$ such that $\tau_\star([e]) = \langle [e], [\mathrm{Ch}^\vee(F)] \rangle$ for any idempotent matrix e over A, then the idempotent Kadison conjecture is true.

- (The Novikov conjecture) Suppose M is a manifold of dimension $4k$, denote $\Gamma := \pi_1(M)$ its fundamental group and $f : M \to B\Gamma$ the associated classifying map. For any cohomology class $\alpha \in H^\star(B\Gamma; \mathbb{Q})$, the **higher-signature**

  $$\sigma_\alpha(M) := \langle L(TM) \cup f^\star \alpha, [M] \rangle \in \mathbb{Q},$$

  where $L(TM)$ is the Hirzebruch L-class of the tangent bundle obtained as a polynomial in the Pontryagin class of M, is a homotopy-invariant.

  Theorem 3.13. *If the Baum-Connes assembly map is rationally injective, then Novikov conjecture holds.*

  Proposition 3.14. *Suppose there exists a dense subalgebra $\mathcal{A}_\Gamma \subseteq C_r^\star(\Gamma)$ preserving K-theory such that the cohomology class $\alpha \in H^\star(B\Gamma; \mathbb{Q})$ extends to a cocyle $[\phi_\alpha] \in HP_0(\mathcal{A}_\Gamma)$. Then, its Chern-Connes pairing together with the Γ-invariant index of the signature operator $\mathrm{Index}_\Gamma(D_{\mathrm{sign}}) \in K_0(C_r^\star(\Gamma))$ recovers the higher-signatures of the manifold*

  $$\sigma_\alpha(M) = \langle \mathrm{Index}_\Gamma(D_{\mathrm{sign}}), [\phi_\alpha] \rangle.$$

Although the pairing provides interesting computation for complex or smooth algebras, the $C^\star$-algebraic case isn't well-behaved.

Theorem 3.15. *For any stable $C^\star$-algebra A (namely $A \otimes_{C^\star} \mathcal{K} \simeq A$), the associated Chern-Connes pairing is identically zero.*

Is there a better homology than periodic cyclic homology? Although periodic cyclic homology provides a computable receptacle of the Chern character coming from K-theory, this construction might be disappointing for some classes of algebras. Other homologies such as entire cyclic homology [Con85, Chapter 7] asymptotic cyclic homology [Pus96] or local cyclic homology [Mey07] have been constructed to overcome this problem and propose alternative solutions in the $C^\star$-algebraic case. As for periodic cyclic homology, they respect various stability and excision properties and are endowed with a Chern character. Moreover, they are preserved by restriction to *smooth* subalgebras and then faithfully reflect the properties of K-theory of $C^\star$-algebras. Here we review some of their main properties.

Entire cyclic homology This cohomology proposes to extend the *dual* Chern character to a broader class of Fredholm modules, although preserving major properties of periodic cyclic homology.

Definition 3.16. *Let A be a Banach algebra. A periodic cyclic cochain $(\phi_{2n})_n$ (resp. $(\phi_{2n+1})_n$) on A is said **entire** if the following serie has an infinite radius of converge*

$$\sum \|\phi_{2n}\| \frac{z^n}{n!} \quad \left(\textit{resp.} \quad \sum \|\phi_{2n+1}\| \frac{z^n}{n!} \right),$$

*where $\|\phi_m\| = \sup_{\|a_i\|\leq 1} |\phi_m(a_0, \cdots, a_m)|$. Denote by $CC^\bullet_\varepsilon(A)$ the periodic cyclic subcomplex made of entire cochains (exo 6), and $CC^\varepsilon_\bullet(A)$ for its homological dual. They compute **entire cyclic homology and cohomology** of the Banach algebra*

$$HC^\varepsilon_\bullet(A) := H_\bullet(CC^\varepsilon_\bullet(A)) \quad \textit{and} \quad HC^\bullet_\varepsilon(A) := H^\bullet(CC^\bullet_\varepsilon(A)).$$

Exercise 6 Show that if ϕ is a periodic cyclic cocycle, one always has $\|b^\vee \phi_m\| \leq (m+2)\|\phi_m\|$ and $\|B^\vee \phi_m\| \leq 2\|\phi_m\|$ for all $m \geq 0$ [Con85, p. 377].

To my knowledge, the original motivation for this definition is the following observation. Given $f : A \to B$ a continuous homomorphism of Banach algebras and τ a continuous trace over B, the pullback $f_\star \tau$ is an entire cocycle over A. In a certain sense, this indicates that the pullback by a continuous algebra homorphism preserves entire cyclic cohomology, i.e. furnishing a candidate for the Grothendieck-Riemann-Roch theorem.

Theorem 3.17 (Puschnigg)**.** *[Pus01] Entire cyclic homology respects the following properties with similar statements for entire cyclic cohomology:*

- *(Morita invariance) Given A and B two Morita equivalent Banach algebras, one has $HC^\varepsilon_\bullet(A) \simeq HC^\varepsilon_\bullet(B)$;*
- *(Excision for involutive Banach algebras) Given A and B two involutive Banach algebras, any continuous extension $0 \to I \to A \to B \to 0$ of Banach algebras with continuous linear section yields a six-term exact sequence*

$$\begin{array}{ccccc} HC^\varepsilon_0(I) & \longrightarrow & HC^\varepsilon_0(A) & \longrightarrow & HC^\varepsilon_0(B) \\ \uparrow & & & & \downarrow \\ HC^\varepsilon_1(B) & \longleftarrow & HC^\varepsilon_1(A) & \longleftarrow & HC^\varepsilon_1(I) \end{array} ;$$

- *(Smooth homotopy invariance) If $f_0, f_1 : A \to B$ are continuous morphisms of Banach algebras, homotopic via a smooth family of continuous morphisms $f_t : A \to B$, then $(f_0)^\bullet = (f_1)^\bullet : HC^\varepsilon_\bullet(B) \longrightarrow HC^\varepsilon_\bullet(A)$. Moreover, if M is a compact manifold, then there exists a natural isomorphism*

$$HC^\varepsilon_\bullet(\mathcal{C}^\infty(M)) \simeq \prod_{n\geq 0} H^{2n+\bullet}_{dR}(M; \mathbb{C}).$$

Remark If we consider a locally convex algebra, the norm of a cochain is not well-defined (because there is no notion of unit-ball in this case). In [Pus96, Chapter 5], Puschnigg proposes a refinement of this cohomology which fits a broader class of algebra. Given a cochain ϕ over locally convex algebra A and a bounded subset $S \subset A$, we set $\|\phi_m\|_S := \sum_{a_i \in S} |\phi_m(a_0, \cdots, a_m)|$. A cochain is called **locally entire** if it is entire over every bounded subset of A. This is equivalent to ask that for any bounded subset $S \subset A$ and $r > 0$, there exists a constant $R > 0$ such that

$$|\phi_{2n}(a_0, \cdots, a_{2n})| \leq R r^{-n} n!, \text{ for any } n \geq 0 \text{ and } a_i \in S.$$

Idem for an odd cochain. Endowing A with its finest locally convex topology, these estimates are well-defined and yields to the definition of **locally-entire** cohomology $HC^\bullet_{\varepsilon,\mathrm{loc}}(A)$. Another equivalent definition is provided by [Pus96] using an analytic version of the Cuntz-Quillen X-complexes [CQ95].

Proposition 3.18. *Let A be a Banach algebra and $\phi = (\phi_{2n})_n$ an entire even periodic cyclic cochain over it. The following assignment is an entire function on the ring of matrices over A*

$$f_\phi : x \longmapsto \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \phi_{2n}(x - 1/2, x, \cdots, x).$$

*Moreover, its restriction to idempotents matrices only depends on the entire cyclic homology class of ϕ and determines an additive map which is called **entire Chern-Connes pairing***

$$\langle -, - \rangle_\varepsilon : K_0(A) \times HC^0_\varepsilon(A) \longrightarrow \mathbb{C}, \quad \langle [e], [\phi] \rangle_\varepsilon := f_\phi(e).$$

The entire cyclic cohomology is then equipped with a Chern-Connes pairing as for periodic cyclic cohomology. If we write ι_ε for the natural inclusion of the entire cyclic complex to the periodic cyclic complex, then $\langle -, - \rangle_\varepsilon = \langle -, \iota_\varepsilon(-) \rangle_{HP}$. The main advantage of this entire theory is that it extends the definition domain of the usual dual Chern character to a broader class of Fredholm modules, namely to any θ-summable spectral triples.

Definition 3.19. *An unbounded Fredholm module $(\mathcal{H}, D, \pi)$ on a locally convex algebra A is said to be **θ-summable** if $\mathrm{Tr}(e^{-D^2}) < \infty$.*

This condition is weaker that the finite-summability according to the next lemma.

Lemma 3.20. *Let $(\mathcal{H}, D, \pi)$ be an unbounded Fredholm-module over a locally convex algebra A. If the associated bounded Fredholm-module is finite-summable, then $(\mathcal{H}, D, \pi)$ is θ-summable.*

Theorem 3.21 (Jaffe-Lesniewski-Osterwalder)**.** [JLO88] *Let A be a unital Banach algebra and $(\mathcal{H}, D, \pi)$ a θ-summable Fredholm module over it. The formula*

$$\phi^\varepsilon_{2n}(a_0, \cdots, a_{2n}) := \int_{\Delta_{2n}} \mathrm{Tr}\left(\gamma e^{-u_0 D^2} \pi(a_0) e^{-u_1 D^2} [F, \pi(a_1)] \cdots e^{-u_n D^2} [F, \pi(a_{2n})]\right) d\Delta_{2n},$$

where $\Delta_m = \{(u_0, \cdots, u_m) \in [0, \infty)^{n+1} \mid u_0 + \cdots + u_m = 1\}$ and γ is the grading morphism, defines an even entire cyclic cocycle $\phi^\varepsilon_D = (\phi^\varepsilon_{2n})_n$ which depends only on the homotopy class of the θ-summable Fredholm module. Moreover, for any idempotent matrix e over A the analytic index of the Fredholm operator $D_e := \pi(e) D \pi(e)$ acting on $\mathcal{H}$ can be expressed as

$$\mathrm{Index}(D_e) = \langle [e], [\phi^\varepsilon_D] \rangle_\varepsilon \in \mathbb{Z}.$$

Remark This entire cyclic cochain ϕ^ε_D is, in particular, a periodic cyclic cochain but two unbounded Fredholm modules defining the same class in K-homology might lead to different periodic cyclic cocycles. It is a consequence of the fact that periodic cyclic homology is *finer* than entire cyclic cohomology and K-theory.

Although entire cyclic cohomology extends the domain of the dual Chern character from finite-summable to θ-summable Fredholm modules, it remains close to periodic cyclic homology. Indeed, these theories coincide for a broad class of algebras. It has been established first in [Pus96] for involutive Banach algebras and generalized in [MS04] for Fréchet topologies.

Theorem 3.22. *Let $\mathcal{F}$ be the smallest class of Fréchet algebras such that:*

- *$\mathbb{C} \in \mathcal{F}$ and $\ell^p \in \mathcal{F}$ for any $p \geq 1$;*
- *$\mathcal{F}$ is closed under smooth homotopy equivalence;*
- *If, in an extension admitting a continuous linear section, two of the algebras belong to $\mathcal{F}$, then so does the third.*

Then for every Fréchet algebra $A \in \mathcal{F}$, the natural inclusion o entire cyclic to periodic cyclic complexes induces an isomorphism $\iota_\varepsilon : HC^\bullet_\varepsilon(A) \xrightarrow{\sim} HP^\bullet(A)$.

Asymptotic cyclic homology The idea behind asymptotic cyclic homology is the following. The reason why entire cyclic homology is a better invariant than periodic cyclic homology is that the requirement for a cocycle ϕ to be entire implies in particular that the norms $\|\phi_n\|$ must decrease exponentially when n increases. The control of the behavior at infinity *rigidifies* the cohomology. This is the motivation behind the notion of asymptotic algebras homomorphisms that are linear map whose curvature tends to zero at infinity.

Definition 3.23. *An **asymptotic morphism** of Fréchet algebras is a bounded continuous family $(f_t)_{t>0}$ of continuous linear maps $f_t : A \to B$ such that for every semi-norm $\|\cdot\|_i$ defining the topology of B, one has*

$$\lim_{t\to\infty} \|f_t(aa') - f_t(a)f_t(a')\|_i = 0.$$

This definition is a reminiscence of the work of Connes and Higson [HC90] on the description of bivariant K-theory groups. They make the use of an intermediate theory called bivariant E-theory which is well described using asymptotic morphisms and deformation. More recently, Meyer proposed a more general architecture for this asymptotic Fréchet category via the notion of bornological vector spaces and extended the definition of asymptotic cyclic homology into the so called local cyclic homology [Mey07]. The major issue of the implementation of this asymptotic category is that an asymptotic algebra homorphism doesn't necessarily induce a periodic cyclic chain-complex. In other words, periodic cyclic homology is not functorial for this class of morphisms. To overcome this problem, the solution is to use another description of periodic cyclic homology due to Cuntz and Quillen. They defined a chain-complex easy to manipulate algebraically, called X-complex, which contains all information encoded in the periodic cyclic homology and even enriches the theory with a *derived functor property* with respect to quasi-free extensions [CQ95]. The refinement of periodic cyclic homology to fit the asymptotic framework will correspond to a refinement of the topology of the X-complex.

Given a locally convex algebra A, its locally convex **tensor algebra** is the algebra $RA := A \oplus (A \otimes_\pi A) \oplus (A \otimes_\pi A \otimes_\pi A) \oplus \cdots$ whose multiplication is given by the tensor product. Let us denote by IA the kernel of the multiplication morphism $RA \to A$. We endow RA with a IA-adic topology and we write $\widehat{RA}$ for the associated completion. The idea of Cuntz and Quillen is that we can recover periodic cyclic homology with a $\mathbb{Z}/2\mathbb{Z}$-graded complex associated to $\widehat{RA}$.

Definition 3.24. *The X**-complex** of a locally convex algebra R is the $\mathbb{Z}/2\mathbb{Z}$-graded complex*

$$X_\bullet(R) := \; R \underset{b}{\overset{B}{\rightleftarrows}} C_1(R)/\overline{b(C_2(R))} \; .$$

where b and B denote the Hochschild's and Connes' differentials.

Theorem 3.25 (Cuntz-Quillen [CQ95])**.** *For every locally convex algebra A, there exists a natural and continuous chain-homotopy equivalence*

$$\mathtt{h} : X_\bullet(\widehat{RA}) \xrightarrow{\sim} \widehat{CC}_\bullet(A).$$

Moreover, for any idempotent matrix $e \in M_r(A)$, there exists an idempotent matrix $\widehat{e} \in M_r(\widehat{RA})$ with $B(\widehat{e}) = 0$ and such that $\mathrm{Ch}([e]) = [\mathtt{h}(\widehat{e})] \in HP_0(A)$. In other words, there exists a Chern character

$$\mathrm{Ch}_X : [e] \in K_0(A) \longrightarrow H_0(X_\bullet(\widehat{RA})) \ni [\widehat{e}]$$

which factorizes the Chern character to periodic cyclic homology over the X-complex.

Lemma 3.26. *Let $f : A \to B$ be an almost multiplicative linear map of Fréchet algebras. The norm of the curvature of the associated linear map $Rf : RA \to RB$ decay exponentially fast with respect to the respective adic valuations.*

Definition 3.27. *For any Fréchet algebra A, denote by $\mathcal{R}A$ the completion of RA for finest locally-Fréchet topology such that any asymptotic morphism $f : A \to B$ induces a continuous morphism of algebras $\mathcal{R}f : \mathcal{R}A \to \mathcal{R}B$. We define* ***asymptotic cyclic homology and cohomology*** *as*

$$HC^\alpha_\bullet(A) := H_\bullet(X_\bullet(\mathcal{R}A)) \quad \textit{and} \quad HC^\bullet_\alpha(A) := H^\bullet(\mathcal{H}om(X_\bullet(\mathcal{R}A), \mathbb{C})).$$

Due to this control at infinity, the asymptotic cyclic homology provides a powerful invariant which is coarser that periodic cyclic homology. Whereas periodic cyclic homology is not stable under continuous homotopy, the next theorem states that asymptotic cyclic homology is stable under restriction to regular enough Fréchet subalgebras.

Definition 3.28. *Let A be a Fréchet algebra. We say that a subalgebra $\mathcal{A} \subseteq A$ is* ***stable under holomorphic calculus*** *if for every element $a \in \mathcal{A}$ and holomorphic function $\mathfrak{f}$ on a neighborhood of the spectrum $\sigma_A(a)$, then $\mathfrak{f}(a)$ still belongs to the subalgebra $\mathcal{A}$.*

Theorem 3.29 (Puschnigg)**.** [Pus96, p. 7.1] *For every admissible (commutative) Fréchet algebra A with subalgebra $\mathcal{A}$ closed under holomorphic calculus, the inclusion $\iota : \mathcal{A} \longrightarrow A$ induces an isomorphism in asymptotic cyclic homology*

$$\iota : HC^\alpha_\bullet(\mathcal{A}) \xrightarrow{\sim} HC^\alpha_\bullet(A).$$

In particular, for every compact manifold M and admissible Fréchet algebras A, the following canonical inclusions yields isomorphisms in asymptotic cyclic homology

$$HC^\alpha_\bullet(\mathcal{C}^\infty(M, A)) \xrightarrow{\sim} HC^\alpha_\bullet(\mathcal{C}^k(M, A)) \xrightarrow{\sim} HC^\alpha_\bullet(\mathcal{C}^0(M, A)).$$

Denote by SA the **suspension algebra** of A, made of of functions in $\mathcal{C}^\infty([0,1], A)$ that vanish at end points. In periodic cyclic homology, it is well known that $HP_\bullet(SA) \simeq HP_{\bullet+1}(A)$ by Bott periodicity but things differ for asymptotic cyclic homology. Indeed, there is not great candidate for Bott element but, after stabilization, one obtains $HC^\alpha_\bullet(S^2A) \simeq HC^\alpha_{\bullet+1}(SA)$ and $HC^\bullet_\alpha(S^2A) \simeq HC^{\bullet+1}_\alpha(SA)$.

Theorem 3.30. *Asymptotic cyclic (co)homology respects the following properties which similar statements for asymptotic cyclic cohomology:*

- *(Morita invariance) Given A and B two Morita equivalent Fréchet algebras, one has $HC^\alpha_\bullet(A) \simeq HC^\alpha_\bullet(B)$;*

- *(Excision for separable $C^\star$-algebras) [Pus96, p. 9.9] Given A and B two separable $C^\star$-algebras, any continuous extension $0 \to I \to A \to B \to 0$ of separable $C^\star$-algebras with continuous linear section yields a six-term exact sequence*

$$\begin{array}{ccccc} HC_0^\alpha(SI) & \longrightarrow & HC_0^\alpha(SA) & \longrightarrow & HC_0^\alpha(SB) \\ \uparrow & & & & \downarrow \\ HC_1^\alpha(SB) & \longleftarrow & HC_1^\alpha(SA) & \longleftarrow & HC_1^\alpha(SI) \end{array} ;$$

- *(Smooth homotopy invariance) [Pus96, p. 6.10] If $f_0, f_1 : A \to B$ are smooth asymptotic morphisms of admissible Fréchet algebras, homotopic via a **continuous** family of continuous morphisms $f_t : A \to B$, then $(f_0)^\bullet = (f_1)^\bullet : HC_\bullet^\alpha(B) \longrightarrow HC_\bullet^\alpha(A)$. Moreover, if X is a locally convex topological space, then there exists a natural isomorphism*

$$HC_\bullet^\alpha(\mathcal{C}_0(X)) \simeq \prod_{n\geq 0} H^{2n+\bullet}(X;\mathbb{C}).$$

Theorem 3.31 (Puschnigg)**.** [Pus96, p. 10.7] *Let $\mathcal{C}$ be the smallest class of Fréchet algebras such that:*

- $\mathbb{C} \in \mathcal{C}$*;*
- *$\mathcal{C}$ is closed under KK-equivalences (in particular if $A \in \mathcal{C}$ and B is a $C^\star$-algebra such that $K_\bullet(A) \simeq K_\bullet(B)$, then $B \in \mathcal{C}$);*
- *If, in an extension admitting a continuous linear section, two of the algebras belong to $\mathcal{C}$, then so does the third.*

Then for every $C^\star$-algebra $A \in \mathcal{C}$, the Chern character of theorem 3.25 extends to the completion $\mathcal{R}A$ and yields rational isomorphisms

$$\mathrm{Ch}_{\alpha,\mathbb{C}} : K_\bullet(A) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} HC_\bullet^\alpha(S^2A) \quad \text{and} \quad \mathrm{Ch}^\vee_{\alpha,\mathbb{C}} : K^\bullet(S^2A) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} HC_\alpha^\bullet(S^2A).$$

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